Elementary Operations in Linear Algebra

Matrix

A rectangular array of numbers. That is numbers are arranged in rows and columns. Subscripts are used to denote the position of an element. By convention, the first subscript denotes the row, and the second denotes the column.

Dimension

The number of rows and columns in a matrix.

Rank

The number of linearly independent rows and columns in a matrix.

Vector

A matrix with a single column or row.

Scalar

A single number, i.e., a matrix with one row and one column.

Square Matrix

The number of rows is the same as the number of columns. The diagonal of a square matrix refers to the diagonal elements starting in the upper left and ending in the bottom right.

All diagonal elements have the same row and column designation, or $A_{i,i}$.

$$A_{K,K} = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \cdots & A_{1,K} \\ A_{2,1} & A_{2,2} & A_{2,3} & \cdots & A_{2,K} \\ A_{3,1} & A_{3,2} & A_{3,3} & \cdots & A_{3,K} \\ \vdots & \vdots & \vdots & & \vdots \\ A_{K,1} & A_{K,2} & A_{K,3} & \cdots & A_{K,K} \end{bmatrix}$$

Some special square matrices are

Symmetric: $A_{i,j} = A_{j,i}$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Diagonal: $A_{i,j} = 0$ for i ... j.

$$A = \begin{bmatrix} 5.1 & 0 & 0 \\ 0 & -2.5 & 0 \\ 0 & 0 & 4.7 \end{bmatrix}$$

Identity: $A_{i,i} = 1$, and $A_{i,j} = 0$ for i ... j. That is the diagonal elements are all 1's and the off diagonal elements are all 0's.

$$[I]_{K,K} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Elementary Operations

Equality

All the elements are the same.

$A_{i,j} = B_{i,j}$;

Addition/subtraction

Combine elements by position, which requires the matrices to have the same dimensions.

For example,

$$A = \begin{bmatrix} 98 & 24 & 42 \\ 35 & 15 & 22 \end{bmatrix}, \qquad B = \begin{bmatrix} 55 & 19 & 44 \\ 43 & 53 & 38 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 153 & 43 & 86 \\ 78 & 68 & 60 \end{bmatrix}$$

Scalar Multiplication

$$cA = \begin{bmatrix} cA_{1,1} & \cdots & cA_{1,K} \\ \vdots & & \vdots \\ cA_{n,1} & \cdots & cA_{n,K} \end{bmatrix}$$

Matrix Multiplication

Place the matrices side by side in the order indicated. The number of columns in the first matrix must equal the number of rows in the second matrix. The dimension of the resultant matrix has the first matrix's row dimension and the second matrix's column designation.

$$[A]_{I,J}\ [B]_{J,L} \quad = \quad [C]_{I,L}, \quad \text{where} \quad C_{i,l} \quad = \quad \sum_{j} A_{i,j} B_{j,l}, \quad \text{so}$$

$$[A]_{n,J}\ [B]_{J,L} \quad = \quad \begin{bmatrix} \sum_{j=1}^{J} A_{1,j}B_{j,1} & \cdots & \sum_{j=1}^{J} A_{1,j}B_{j.L} \\ \vdots & \cdots & \vdots \\ \sum_{j=1}^{K} A_{n,j}B_{j,1} & \cdots & \sum_{j=1}^{K} A_{n,j}B_{j,L} \end{bmatrix},$$

$$= \begin{bmatrix} C_{1,1} & \cdots & C_{1,L} \\ \vdots & \cdots & \vdots \\ C_{n,1} & \cdots & C_{n,L} \end{bmatrix}$$

For example:

$$[A]_{2,3}\ [B]_{3,2} \quad = \quad [C]_{2,2}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \quad = \quad \begin{bmatrix} 4 & 14 \\ 10 & 10 \end{bmatrix}$$

The order is crucial. For example:

$$[B]_{3,2}[A]_{2,3} = [C]_{2,4}$$

$$\begin{bmatrix} 0 & 6 \\ 1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & 3 \end{bmatrix} \quad = \quad \begin{bmatrix} -6 & 24 & 18 \\ 0 & 4 & 5 \\ -2 & 16 & 16 \end{bmatrix}$$

But suppose B is now a 3x4 matrix:

$$[A]_{2,3}\,[B]_{3,4} \quad = \quad [C]_{2,4}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & 6 & 1 & 5 \\ 1 & 1 & 2 & 7 \\ 2 & 4 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 14 & 9 & 11 \\ 10 & 10 & 19 & 32 \end{bmatrix}$$

But the reverse, $[B]_{3,4}\,[A]_{2,3}$ is not defined.

Vector Multiplication

Either a scalar or a matrix is the result.

$[A]_{1,n}\,[B]_{n,1} = [C]_{1,1}$

$[A]_{n,1}\,[B]_{1,n} = [C]_{n,n}$

For example

$$[A]_{1,3}\,[B]_{3,1} = [C]_{1,1}$$

$$\begin{bmatrix} 5, & 2, & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 33$$

$$[B]_{3,1}\,[A]_{1,3} = [C]_{3,3}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 5, & 2 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 8 \\ 10 & 4 & 16 \\ 15 & 6 & 24 \end{bmatrix}$$

Transpose

Interchange the rows and columns, denoted by a T or N superscript.

$$[A_{i,j}]' \quad = \quad A_{j,i}$$

$$\begin{bmatrix} 3 & 1 \\ -4 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix}, \quad \textit{and}$$

$$[A']' \quad = \quad A$$

Determinant

A scalar that is the sum of selected products of the elements of a square matrix, usually written as |A|.

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

For a 3x3 matrix A:

$$\left|A_{3,3}\right| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gfb - ceg - fha - idb$$

If the determinant of a matrix = 0, then at least one row (or column) is a linear combination of the other rows (or columns).

Minor

The determinant of a matrix formed by dropping the ith row and jth column.

$$|M_{i,j}|$$

Cofactor

A signed minor

$$m_{i,j} = (-1)^{i+j} \left|M_{i,j}\right|$$

Inverse

The process of division does not exist in linear algebra. It is replaced with the process of multiplication by a matrix called the inverse. The inverse of a matrix A is a matrix whose product with A is the identity matrix.

$$A\,A^{-1} = I, \text{ and } A^{-1}\,A = I$$

This type of inverse is a square matrix and is unique.

The inverse can be expressed in terms of cofactors. The usefulness of this way of expressing the inverse will be seen when we discuss the problem of multicollinearity.

$$[A]^{-1}_{3,3} = \frac{1}{|A|} \begin{bmatrix} \mu_{1,1} & \mu_{1,2} & \mu_{1,3} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} \end{bmatrix}$$

Sum of the squares and cross products.

Let the matrix X have the form:

$$[X]_{nK} = \begin{bmatrix} 1 & x_{1,2} & \cdots & x_{1,K} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n,2} & \cdots & x_{nK} \end{bmatrix} = , \quad \text{so}$$

$$X'X = \begin{bmatrix} 1 & \cdots & 1 \\ x_{1,2} & \cdots & x_{n2} \\ \vdots & \cdots & \vdots \\ x_{i,K} & \cdots & x_{nK} \end{bmatrix} \begin{bmatrix} 1 & x_{1,2} & \cdots & x_{1,K} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n2} & \cdots & x_{n,K} \end{bmatrix},$$

$$= \begin{bmatrix} n & \sum_{i=1}^{n} x_{i,2} & \cdots & \sum_{i=1}^{n} x_{i,K} \\ \sum_{i=1}^{n} x_{i,2} & \sum_{i=1}^{n} x_{i,2}^2 & \cdots & \sum_{i=1}^{n} x_{i,2}x_{i,K} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=1}^{n} x_{i,K} & \sum_{i=1}^{n} x_{i,K}x_{i,2} & \cdots & \sum_{i=1}^{n} x_{i,K}^2 \end{bmatrix}$$

The sum of the squares and cross products is a symmetric, positive definite matrix. The latter is useful in analyzing the second order conditions for a minimum.

Trace

The sum of the diagonal elements. Only applies to square matrices.

Idempotent matrix

A matrix pre or post multiplied by itself results in the same matrix.

A A = A, so it only applies to square matrices.

The trace of an idempotent matrix equals its rank.

For example:

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix}\begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 4 & 8 & 12 \\ -3 & -6 & -9 \end{bmatrix}$$

An idempotent matrix is useful in regression analysis. Suppose there is a matrix $[X]_{n,K}$. Then the product $[X]_{K,n}' [X]_{n,K}$ is a KxK matrix.

Let $A = X(X'X)^{-1}X'$. Then:

$$X(X'X)^{-1}X'X(X'X)^{-1}X' = X(X'X)^{-1}X'$$